Inverse functional relation on the Potts model

1982 J. Phys. A: Math. Gen. 15 2241

(http://iopscience.iop.org/0305-4470/15/7/034)

J. Phys. A: Math. Gen. 15 (1982) 2241–2257. Printed in Great Britain

Inverse functional relation on the Potts model

M T Jaekel† and J M Maillard‡

† Laboratoire de Physique Théorique de L'Ecole Normale Supérieure§
‡ Service de Physique due Solides et Résonances Magnétiques, CEN Saclay, Gif-sur-Yvette, France

Received 20 November 1981, in final form 8 February 1982

Abstract. The two-dimensional anisotropic Potts model at all temperatures satisfies a local inverse relation leading to an inverse functional equation on the transfer matrix. The pertinence of the related functional equation for the partition function is discussed and verified perturbatively, by introducing a diagrammatic expansion corresponding to a partial resummation on one parameter of the model. Possible exploitations of the inverse relation in order to determine the partition functions are sketched. This indicates that the inverse relation can be used without referring to a star-triangle relation.

1. Introduction

Ten years ago R J Baxter emphasised the importance of a generalised star-triangle relation for exactly soluble models in statistical mechanics (Baxter 1972, 1973a, 1978). More recently, this relation has also proved to be useful for some models of particle physics or field theory (S-matrix factorisation: Berg and Weisz (1978), Zamolodchikov (1979), Zamolodchikov and Zamolodchikov (1979); quantum inverse scattering: Sklyanin *et al* (1979), Kulish and Sklyanin (1979)). It has been shown that, from the star-triangle relation, one can deduce the commutativity of a family of transfer matrices. A linked concept, the Bethe ansatz (Bethe 1931), then enables transfer matrices to be diagonalised and the partition function to be calculated (Baxter 1972, 1973a). As a byproduct of these notions a new relationship has occurred which is called the inverse relation. It is seen by Baxter (1980b), Stroganov (1979) and Schultz (1981) as a short cut and a practical tool for calculating the partition function. (A formal identification exists between this relation and the unitarity relation for the S-matrix, as has been remarked by Zamolodchikov (1979).) In this context, star-triangle and inverse relations seem to be linked. Gaudin (1979) has even observed that these two relations express the relations which generate the group of permutations (Coxeter and Moser 1972). Nonetheless, the question remains of whether the inverse relation can be used independently of the star-triangle relation. Furthermore, the inverse relation concept is not clear as regards the following point: if this local relation leads in a clear way to an inverse relation concerning an object with a global character, the transfer matrix, does it lead to a functional relation for the partition function?

The purpose of this paper is to reply positively to both these questions: we shall exhibit the matrix inverse relation and the associated functional relation for the anisotropic Potts model at all temperatures, and not only at the critical temperature

§ Laboratoire Propre de Centre National de la Recherche Scientifique associé à l'Ecole Normale Supérieure et à l'Université de Paris Sud. Postal address: 24, rue Lhomond, 75231 Paris Cedex 05, France.

T_c where it is known that the star-triangle relation occurs. In § 2 we discuss the inverse relation concept. Firstly, we recall how the local relation implies the global matrix relation sought. Secondly, we write and discuss the functional relation to be verified by the partition function. In § 3, since an analytical knowledge of the partition function is not yet available, we verify the reality of the preceding functional relation using a diagrammatic expansion. As the usual expansions are insufficient, we introduce a new diagrammatic expansion which corresponds to a partial resummation on one of the two variables of the anisotropic model. In § 4 the problem is looked at again finally from another point of view: the exploitation of the inverse relation which we consider, now, to be valid.

2. Inverse relation for the Potts model

We first sum up the q-state, scalar, two-dimensional anisotropic Potts model for a square lattice (see figures 1 and 2) (Potts 1952, Baxter *et al* 1978). The following conventions are adopted here: if σ_i and σ_j belonging to Z_q are in the same state the statistical weight associated with this vertical bond will be c, if not it will be $+1$; if σ_j and σ_k (horizontal bonds) are in the same state it will be b, if not it will be $+1$.

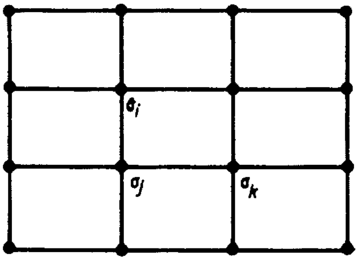


Figure 1.

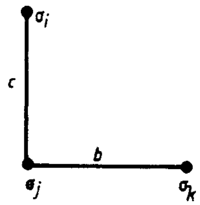


Figure 2.

The partition function is therefore

$$Z = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} c^{\delta_{\sigma_i,\sigma_j}} \prod_{\langle jk \rangle} b^{\delta_{\sigma_j,\sigma_k}}$$

where the products are to be taken over all the horizontal and vertical bonds and the sum is to be taken over all the configurations of spins.

2.1. Matrix inverse relation

Let us rapidly find for the Potts model the inverse relation verified by the local statistical weights and the global inverse relation on the transfer matrix. Consider the algebraic method presented by Temperley and Lieb (1971). Let $A, B, C, \ldots$ be the N sites along a line and $A', B', C', \ldots$ be the N sites on the line above as shown in figure 3. Reintroduce the notations of Baxter *et al* (1978): $\sigma_A, \sigma_B, \sigma_C, \ldots$ designate spins at sites $A, B, C, \ldots$ and $\sigma_{A'}, \sigma_{B'}, \sigma_{C'}, \ldots$ at sites $A', B', C', \ldots$; V_A (resp. W_{AB}) are operators corresponding to adding the vertical edge AA' (resp. the horizontal edge AB), and similarly W_{BC}, V_B, etc.... They can be written as $q^N \times q^N$ matrices

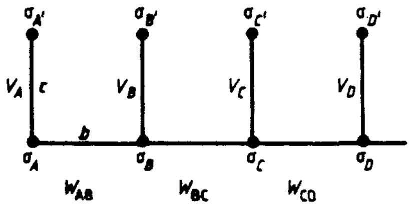


Figure 3.

with lines indexed by $\sigma' = \{\sigma_{A'}, \sigma_{B'}, \ldots\}$ and columns by $\sigma = \{\sigma_A, \sigma_B, \ldots\}$. The corresponding elements are then

$$(V_A)_{\sigma',\sigma} = [1+(c-1)\delta(\sigma_A, \sigma'_A)]\delta(\sigma'_B, \sigma_B)\delta(\sigma'_C, \sigma_C)\ldots$$
$$(W_{AB})_{\sigma',\sigma} = [1+(b-1)\delta(\sigma_A, \sigma_B)]\delta(\sigma'_B, \sigma_B)\ldots \tag{1}$$

(δ is the usual Kronecker symbol) and in a similar way for V_B, $W_{BC}, \ldots$. It is easy to verify that

$$V_A(c)\cdot V_A(2-q-c) = (c-1)(1-q-c)I \tag{2}$$

$$W_{AB}(b)\cdot W_{AB}(1/b) = I \tag{3}$$

where I is the identity matrix.

The transfer matrix that sends the line underneath to the line above is a product of two matrices: $T = T_1 \cdot T_2 = (\ldots V_C V_B V_A)(\ldots W_{CD} W_{BC} W_{AB})$, and therefore

$$T_1(c)\cdot T_1(2-q-c) = (c-1)^N(1-q-c)^N I \tag{4}$$

$$T_2(b)\cdot T_2(1/b) = I. \tag{5}$$

Instead of T, introduce $\tilde{T} = T_2^{1/2}\cdot T_1 \cdot T_2^{1/2}$ which then satisfies

$$\tilde{T}(b,c)\cdot\tilde{T}(1/b, 2-q-c) = [(c-1)(1-q-c)]^N I. \tag{6}$$

One can see that simple local relations such as (2) and (3) (which can be represented graphically as in figure 4) lead to a similar relation for a global object such as $\tilde{T}(b,c)$.

Figure 4.

2.2. Functional equation for the partition function per site

We shall now look at what the consequence of relation (6) might be on the partition function of the model. Designate the number of lines by M. With the usual periodic conditions, the partition function is equal to the trace of the products of M transfer matrices $\tilde{T}$: $Z = \mathrm{Tr}(\tilde{T})^M$. At the thermodynamic limit, the Z notation will be used for the partition function per site

$$\lim_{\substack{M\to\infty\\ N\to\infty}} Z^{1/MN}$$

and thus Z will be equal to the Nth root of the largest eigenvalue of the transfer matrix. Let $|\Omega\rangle$ be the eigenvector corresponding to the largest eigenvalue $\lambda(b, c)$ of $\tilde{T}(b, c)$. Equation (6) leads to

$$\begin{aligned}\tilde{T}(b, c)\cdot\tilde{T}(1/b, 2-q-c)|\Omega\rangle\\ &= \tilde{T}(1/b, 2-q-c)\tilde{T}(b, c)|\Omega\rangle\\ &= \lambda(b, c)\tilde{T}(1/b, 2-q-c)|\Omega\rangle\\ &= (c-1)^N(1-q-c)^N|\Omega\rangle\end{aligned} \tag{7}$$

and empirically

$$= \lambda(b,\ c)\lambda(1/b, 2-q-c)|\Omega\rangle \tag{8}$$

and in consequence

$$Z(b, c)\cdot Z(1/b, 2-q-c) = (c-1)(1-q-c). \tag{9}$$

Care is necessary, however. Let us study equalities (7) and (8). The equalities (7) need no further discussion, as they arise from the previously established hypotheses: $\tilde{T}(b, c)$ and $\tilde{T}(1/b, 2-q-c)$ are inverse matrices up to the multiplicative factor $(c-1)^N(1-q-c)^N$ and therefore commute; $|\Omega\rangle$ is an eigenvector of $\tilde{T}(b, c)$ with eigenvalue $\lambda(b, c)$. From equalities (7), $|\Omega\rangle$ is also an eigenvector of $\tilde{T}(1/b, 2-q-c)$ with eigenvalue $[(c-1)^N(1-q-c)^N]/\lambda(b, c)$ deduced. More can be deduced from equality (8): this equality confirms that this last eigenvalue is merely $\lambda(1/b, 2-q-c)$; in other words, that one is dealing with the same function for different arguments $1/b$ and $2-q-c$. What is meant by saying the *same* function? This does not mean the partition function with the new values for the parameters. Let us consider, for example, the two-dimensional Ising model. The inverse relation is then $(b, c)\to(1/b, -c)$ or, with the usual notations for the coupling constant, $(K_1, K_2)\to(-K_1, K_2+i\pi/2)$. The partition function for the new parameters, with the usual periodic conditions, is defined as

$$\begin{aligned}Z(-K_1, K_2+i\pi/2) &= \sum_{\{\sigma\}}\prod_{\langle ij\rangle} e^{-K_1\sigma_i\sigma_j}\prod_{\langle jk\rangle} e^{(K_2+i\pi/2)\sigma_j\sigma_k}\\ &= \sum_{\{\sigma\}}\prod_{\langle ij\rangle} e^{-K_1\sigma_i\sigma_j}\prod_{\langle jk\rangle}(i\sigma_j\sigma_k)\, e^{K_2\sigma_j\sigma_k}\end{aligned}$$

which is simply (up to a multiplicative factor $\pm i$) $Z(-K_1, K_2) = Z(K_1, K_2)$. Equation (9) would then indicate that Z^2 is a known function. It is easy to verify that this is not the case.

Accordingly, one can see that equation (7) must relate the largest eigenvalue of $\tilde{T}(b, c)$ to the smallest eigenvalue of $\tilde{T}(1/b, 2-q-c)$. From the situation of finite N it is natural to consider that relation (9) takes place between the function $\lambda(b, c)$, or the partition function $Z(b, c)$, and an analytical continuation of this function at $(1/b, 2-q-c)$ (Baxter 1980b). One can verify that this interpretation is correct in the thermodynamic limit for the simple case of the one-dimensional Ising model (with magnetic field). For more delicate cases Stroganov (1979) and, especially, Baxter (1980b) seem to justify this fact by considering the complete integrability of the models. In these cases a family of commuting transfer matrices exists, with a θ parameter describing an algebraic curve. There is therefore an analytical path which leads from the eigenvalue of the transfer matrix at the point θ, $\lambda(\theta)$, to its analytical continuation at the inverse point $-\theta$. In practice, Baxter writes such a relation only once he has established the complete integrability from the generalised star-triangle relation, and once he has been able to uniformise this relation. However, examples exist of models possessing an inverse relation for which a star-triangle or an equivalent relation is not yet known (rubber bands and percolation models, two-dimensional non-planar or with-field models, three- and even higher-dimensional Potts models (Jaekel and Maillard (1982)).

In particular, that is the case for the Potts model at all temperatures. There, no parametrisation exists, as in the Ising case, to reduce the set of parameters to a single variable: in fact, this problem seems to truly involve two complex variables, and any justification of (9) seems to be more complicated. The best way to be convinced of the relation (9) is to show it directly using empirical methods. As an analytical solution for the partition function in the Potts model is not yet available, we shall use expansions. In this way we shall have shown, up to a certain order, that equation (9) holds for the Potts model at all temperatures (and not only at the critical temperature T_c where the model is known to be completely integrable). This will thus simultaneously provide an example of the fact that the generalised star-triangle relation and the inverse relation are not mutually dependent.

To conclude this section let us remark that this discussion is not limited to the largest eigenvalue. If we let $\lambda_a(b, c)$ be a generic eigenvalue, with the same reserve, the functional equation can be written as

$$\lambda_\alpha(b, c)\lambda_\alpha(1/b, 2-q-c) = (c-1)^N(1-q-c)^N.$$

For instance, if $\lambda_1(b, c)$ designates the second-largest eigenvalue, one has the functional equation

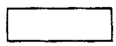
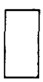
$$\frac{\lambda(b, c)}{\lambda_1(b, c)}\,\frac{\lambda(1/b, 2-q-c)}{\lambda_1(1/b, 2-q-c)} = 1.$$

This is easy to verify in the special Ising case using the interface energy expression (Watson 1977).

3. Diagrammatic expansion for the Potts model

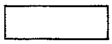
A standard low-temperature expansion such as that of Kihara *et al* (1954) $1/b \to 0, 1/c \to 0$) gives

$$Z(b, c) = bc\left[1 + \left(\frac{q-1}{b^2c^2}\right) + \cdots\right] = bc\;\Lambda(b, c).$$

The simplest and most elegant way of obtaining this expansion is to use a method, explained for example by Ginsparg *et al* (1980), which is based on exploiting the characters of the group $\boldsymbol{Z}_q$. For instance, the first term $(q-1)/b^2c^2$ corresponds to the diagram , the following term $(q-1)/b^4c^2$ to , the term $(q-1)/b^2c^4$ to , etc....

In order to use relation (9) it is necessary to have available the expansion of the same analytical function about a point and about the inverse of the point. In the preceding expansion we see that if $1/c \sim 0$ is effectively stable by $c \to 2-q-c$, on the other hand $1/b \sim 0$ becomes $1/b \sim \infty$. The standard low-temperature expansion is thus not stable under the inverse transformation. It is therefore not possible to use it in equation (9).

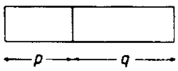
The type of expansion one needs to exploit (9) is, for example, one with small $1/c$ values and arbitrary values for b. Fortunately, it is possible to obtain this new type of diagrammatic expansion, as we shall now show. We wish to obtain an expansion in powers of $1/c$, so let us look at all the terms in $1/c^2$:

 $\frac{q-1}{b^2c^2}$  $\frac{q-1}{b^4c^2}$  $\frac{q-1}{b^6c^2}$ $\ldots$.

It is easy to sum over all these diagrams, which compose a geometrical series, to obtain

$$\left(\frac{q-1}{c^2}\right)\left(\frac{1}{b^2-1}\right).$$

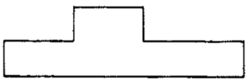
The function $f(b)=1/(b^2-1)$ analytically extends the preceding series outside its convergence disc 1 $|b|<1$. Let us now look at terms in $1/c^3$. From the diagrammatic point of view they are of the form



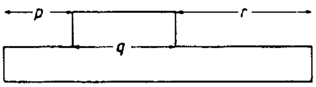
where $p, q \geqslant 1$. A straightforward summation over all these terms gives

$$\frac{1}{c^3}(q-1)(q-2)\left(\frac{1}{b^2-1}\right)^2.$$

It is then natural to represent the series diagrammatically by one of its elements, thus not having to take the length variation of the different sections explicitly into consideration. For example, at fourth order



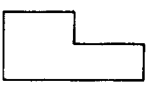
will represent the diagrams



where $p, q, r \geqslant 1$. The corresponding contribution is

$$\frac{q-1}{c^4}\left(\frac{1}{b^2-1}\right)^3.$$

These diagrams differ for example from the following ones:



of contribution

$$\frac{q-1}{c^4}\left(\frac{1}{b^2-1}\right)^2.$$

One can convince oneself that this type of partial resummation in b extends to all orders $1/c^n$. It therefore gives the type of expansion we need. Let us then insert this expansion into relation (9), which we now write in the form

$$\ln \Lambda(b, c)+\ln \Lambda\left(\frac{1}{b}, 2-q-c\right)=\ln\left(\frac{(c-1)(1-q-c)}{c(2-q-c)}\right). \tag{10}$$

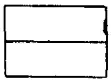
The diagrammatics for $\ln\Lambda(b, c)$ differs from that for $\Lambda(b, c)$ in the counting of disconnected terms and of course in the initial term $+1$. Thus, up to second order, the left-hand term in (10) is

$$\ln \Lambda(b, c)=\frac{q-1}{c^2}\left(\frac{1}{b^2-1}\right)+\cdots$$

$$\ln \Lambda\left(\frac{1}{b}, 2-q-c\right)=\frac{q-1}{c^2}\left(\frac{-b^2}{b^2-1}\right)+\cdots$$

and the right-hand term in (10) is equivalent to

$$\ln\left(1+\frac{q-2}{c}+\frac{1-q}{c^2}\right)-\ln\left(1+\frac{q-2}{c}\right)=\frac{1-q}{c^2}+\cdots.$$

This shows how the two terms in $1/c^2$ combine together on the left-hand side of equality (10) to give a term independent of b. We have verified equality (10) up to order $1/c^5$ in appendix 1. It is instructive to examine the details of these calculations. In particular, relation (10) imposes strong restraints on the expansions. For instance, one can see, in contrast with the Ising model ($q=2$) for which only $1/(b^2-1)^n$-type singularities appear, that, in the general case of the Potts model, singularities at the nth root of unity occur: for the coefficients of $1/c^{2r}$ and $1/c^{2r+1}$, singularities at $b^{r_1}=1$ occur for all r_1 integers $\leqslant r+1$. For example, the term 

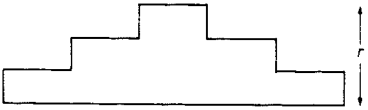
$$\text{contributes}\ \frac{(q-1)(q-2)}{c^4}\left(\frac{1}{b^3-1}\right).$$

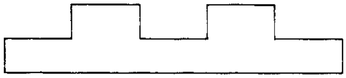
It is then interesting to see how all the singularities from the two expansions combine together and successively eliminate, starting from the strongest one. To conclude this section we have also verified, in appendix 2, the inverse relation at all orders $1/c^n$ on the leading diagrams in q.

4. Uses of the functional equation

4.1. Diagrammatic approach

Let us look at the expansion and the inverse relation, this time from a different point of view: we shall use them to try to resolve the partition function. Let us describe all the terms up to a given order $1/c^{2r}$. It is easy to describe terms having a single singularity $b^2=1$. The largest-order pole is obtained from the following diagrams:



or 

or  or  ....

Of course smaller-order poles occur, but there is only a single diagram which gives the simple pole:

$$\frac{q-1}{c^{2r}(b^2-1)}$$

It seems possible to regroup all these terms in a simple expression

$$\frac{1}{c^{2r}}\frac{P_r(b^2)}{(b^2-1)^{2r-1}}$$

where P_r is a polynomial of order $2(r-1)$ with rational coefficients. Baxter (1980b) took advantage of this compact form for the Ising model where effectively only the singularity $b^2=1$ occurs. The inverse relation becomes $(b,c)\to(1/b,-c)$ and the functional relation

$$\ln\Lambda(b,c)+\ln\Lambda(1/b,-c)=\ln(1-1/c^2).$$

Introducing

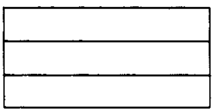
$$\ln\Lambda(b,c)=\sum_{r=1}^{\infty}\frac{1}{c^{2r}}\frac{P_r(b^2)}{(b^2-1)^{2r-1}}$$

one has for P_r

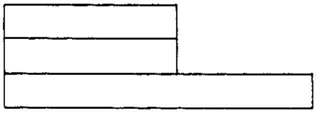
$$\frac{P_r(b^2)}{(b^2-1)^{r-1}}+\left(\frac{-b^2}{b^2-1}\right)^{2r-1}P_r\left(\frac{1}{b^2}\right)=-\frac{1}{r}. \tag{11}$$

This relation indicates that, if the first $r-1$ coefficients of P_r are known, the polynomial P_r can be determined completely. If one now assumes in a given recurrence that one knows $P_1\ldots P_{r-1}$, then from the symmetry $\ln\Lambda(b,c)=\ln\Lambda(c,b)$ one can determine the preceding $r-1$ coefficients of P_r. This is true for all r values, and one can therefore use only the inverse and symmetry relations to calculate the successive terms of the partition function in a unique way.

Let us come back to the general case of the Potts model. As far as the other singularities are concerned, we have already seen that they are of the $b^{r_1}=1$ $(r_1\leq r+1)$ type. The order of these poles decreases for increasing r, eventually becoming a simple pole for $r_1=r+1$ with diagrams such as

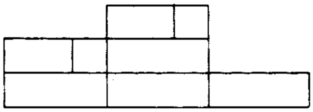


$$\frac{(q-1)(q-2)^{r-1}}{c^{2r}}\left(\frac{1}{b^{r+1}-1}\right)$$



$$\frac{(q-1)(q-2)^{r-1}}{c^{2r}}\left(\frac{1}{b^{r+1}-1}\right)\left(\frac{1}{b^2-1}\right).$$

There is a large number of terms, mixing all these singularities, as for example



$$P(q)\left(\frac{1}{b^4-1}\right)^2\left(\frac{1}{b^3-1}\right)^2\left(\frac{1}{b^2-1}\right).$$

For this reason, any compact form for the $1/c^{2r}$ coefficient will depend on many more coefficients than in the Ising case: the information obtained from the symmetry is not enough to determine all of them. These considerations apply to the $1/c^{2r+1}$ coefficients. Thus, a recurrence of the type Baxter used for the Ising case seems to fail here. However, we have not used all the information at our disposal. For example, all the relations one writes must be verified for all q values. The $1/c^{2r}$ coefficient is a rational function in b^2 and a polynomial in q with rational coefficients. More detailed studies of the q dependence could enable the various diagrams to be investigated in a more selective manner. For instance, it has already been seen that diagrams possessing the maximum number of loops automatically satisfy the inverse relation (appendix 2).

4.2. Analytical approach

Rather than determining the partition function perturbatively, it is possible to imagine determining it globally from the preceding functional equation and from the symmetry equation

$$Z(b,c)Z(1/b,2-q-c)=-(c-1)(c+q-1) \qquad \text{(I)} \qquad (12)$$

$$Z(b,c)=Z(c,b) \qquad \text{(S)}. \qquad (13)$$

These two equations are to be viewed as equations between the partition function and its analytical continuation (though the analytical situation might not be clear with this set of variables; see appendix 3). These equations are very like the functional equations for automorphic functions. From this point of view, the partition function is a sort of generalisation to several complex variables of automorphic functions. Indeed, the two transformations, inverse and symmetry (I and S), generate an infinite group G. One can see that this group G satisfies the exact sequence

$$0\to \boldsymbol{Z}\to G\to \boldsymbol{Z}_2\to 0$$

and is compatible with the 'automorphic' factors appearing in equations (12) and (13). G also possesses a special element

$$(\mathrm{SI})^2:(b,c)\to\left(2-q-\frac{1}{b},\ \frac{1}{2-q-c}\right)$$

transforming b and c separately. The fixed points of these two transformations are

$$q_\pm = 1-\tfrac{1}{2}q\pm\tfrac{1}{2}\sqrt{q(q-4)}. \qquad (14)$$

If we now consider new variables

$$x=(b-q_+)/(b-q_-) \qquad y=(c-q_+)/(c-q_-)$$

the transformations I and S take the simple form

$$\text{I}:(x, y) \to (1/x, q_+^2/y) \qquad \text{S}: (x, y) \to (y, x).$$

It can be seen that a knowledge of the partition function at some point (b_0, c_0) or (x_0, y_0) determines the function at an infinite number of points. It is known that functional equations of that type do not determine the function in a unique way (the function can be chosen arbitrarily on the fundamental domain). Such a system of equations is nonetheless restrictive. It can be rewritten with new variables x and y:

$$Z(x, y) \cdot Z\left(\frac{1}{x}, \frac{q_+^2}{y}\right) = -qq_+ \frac{(1+x/q_+)(1+1/q_+x)}{(1-x)(1-1/x)} \tag{15}$$

$$Z(x, y) = Z(y, x). \tag{16}$$

Let us first consider the special case of the critical temperature $(b-1)(c-1)=q$ or $xy=-q_+$. The problem being reduced to one with a single complex variable, it is then possible to find the partition function directly from equations (15) and (16), and a maximal analyticity assumption. As we are at $T=T_c$, which is a curve $(xy=-q_+)$ stable under the group G, the symmetry (S) is expressed by $x \to -q_+/x$ and the inverse (I) by $x \to 1/x$. Introducing the notation $f_g(x)=f(g(x))$, equations (15) and (16) become

$$Z \cdot Z_{\text{I}} = -qq_+ L \cdot L_I \qquad Z = Z_{\text{S}}$$

with

$$L(x) = \frac{1+x/q_+}{1-x}.$$

Letting

$$A = \sum_{n=0}^{\infty} L_{(\text{SI})^{2n}} \qquad A(x) = \frac{\Pi_{n=0}^{\infty}(1+q_+^{2n-1}x)}{\Pi_{n=0}^{\infty}(1-q_+^{2n}x)}, \tag{17}$$

A satisfies $A = L \cdot A_{(\text{SI})^2}$. One can see that

$$Z\left(x, -\frac{q_+}{x}\right) = \sqrt{-qq_+}\frac{A \cdot A_{\text{S}}}{A_{\text{SI}}A_{\text{SIS}}}(x) = \sqrt{-qq_+}\frac{A(x)A(-q_+/x)}{A(-q_+x)A(q_+^2/x)} \tag{18}$$

satisfies (15) and (16) (Jaekel and Maillard 1981).

The physical difference between $q<4$ and $q>4$ appears here to be natural. Recalling equation (14), expression (18) only has a meaning for $q>4$ or $q<0$ where the expression converges. When $0<q<4$ the transformation $(\text{SI})^{2n}: x \to q_+^{2n} x$ does not converge to a fixed point (q_+ lies on the unit circle $|q_+|=1$). Nonetheless, it is known that the partition function can be obtained at T_c for any values of q (see (46*b*) and (46*c*) in Baxter *et al* (1978)†). In the case $q<4$ or $q=4$ expression (46*c*) of Baxter *et al* can be obtained using the Malmsten representation of $\ln\Gamma(z)$ (Magnus *et al* 1972). So, even in the case $q \leqslant 4$ where the infinite product (18) has no meaning, it is possible to obtain the desired result by an analytical continuation in the q variable. Let us notice that this gives a result even for the $q=0, 1, 2, 3, 4$ cases where not only $|q_+|=1$, but also G is a finite group. It is interesting to note that in the Ising case the group G can be seen either as a finite group with the preceding rational uniformisation

† The correspondence with the variables used in Baxter *et al* is as follows: $-q_+ = e^{2\lambda}$, $x = e^{2\alpha_1}$, $y = e^{2\alpha_2}$.

or as an infinite group with the usual elliptical uniformisation (see appendix 3)†. Expression (18) has been obtained for the Potts model without using the known connection with the six-vertex model (Baxter *et al* 1978). This connection would have led us to use, more or less directly, the star-triangle relation or the Bethe ansatz. This approach sheds some light on the fact that Eulerian products often occur in exactly soluble models, as Baxter *et al* (1975, 1976) have already noticed. The 'natural parameters' they were led to introduce can be seen to be the variables for which the action of G becomes multiplicative. Moreover, equation (18) shows that the inverse relation and the symmetry relation can also lead to elliptical curves.

Coming back now to the general case of all temperatures, because of the presence of two complex variables it is necessary to state more information about the function. For example, if some maximal analyticity hypothesis was enough for the case of one complex variable, the nature of the analyticity of the partition function is here more difficult to state. The diagrammatic expansion just provides analyticity in b for $1/c$ small and in c for $1/b$ small, and appendix 3, for instance, shows that the partition function might not be uniform for arbitrary b and c. Here, the information at our disposal for the Potts model is the following: the partition function is known in some particular cases b or $c=1$ (one-dimensional model), $T=T_c$ (Baxter *et al* 1978), $q=1$ (obvious), and in certain neighbourhoods thanks to expansions; high- or low-temperature expansions (Kihara *et al* 1954), large-q expansions (Ginsparg *et al* 1980). We also have the exact latent heat at $T=T_c$ (Baxter 1973b) (it is related to the spontaneous polarisation of the six-vertex model, which surprisingly does not depend on the anisotropy of the model at $T=T_c$ but only on q_+). Finally it can be verified that the Kramers–Wannier duality (Kramers and Wannier 1941) is consistent with the group G. From this point of view, the problem still seems to be difficult, as one has to deal with functions of several complex variables. To illustrate this one can (for $q>4$) envisage making the following change of variables and functions:

$$\lambda = xy \qquad \mu = \frac{x}{y} \qquad Z(x, y) = \sqrt{-qq_+}\,\frac{A(x)A(y)}{A(q_+^2/x)A(q_+^2/y)}\tilde{Z}(\lambda, \mu).$$

From (15) and (16) we can see that $\tilde{Z}$ satisfies the symmetry and inverse relations deduced:

$$\tilde{Z}(\lambda, \mu) = \tilde{Z}(\lambda, 1/\mu) \qquad \tilde{Z}(\lambda, \mu)\cdot\tilde{Z}(q_+^2/\lambda, 1/q_+^2\mu) = 1$$

and thus $\tilde{Z}(\lambda, \mu) = \tilde{Z}(\lambda, q_+^4\mu)$.

If $\tilde{Z}$ were regular around the point (λ, ∞), then it would be possible to write $\tilde{Z}(\lambda, \mu) = \tilde{Z}(\lambda, \infty)$, and thus f and g would exist such that $Z(x, y) = f(x)f(y)g(xy)$. But this is ruled out, as can be seen on a large-q expansion for instance‡.

5. Conclusion

We have shown, for the Potts model, the existence of a matrix relation, the inverse relation, and of a corresponding functional equation for the partition function. In

† Since completing this work we have become aware of a preprint by Baxter which relates the critical hard-hexagon model to the critical Potts model for $q=(3+\sqrt{5})/2$. It is amusing to remark that for this particular value of q the group G is finite of order 10.

‡ We have computed the large-q expansion of $\ln Z$ up to sixth order in $\sqrt{-q_+}$ and anisotropic in $x/\sqrt{-q_+}$ and $y/\sqrt{-q_+}$ (to be published). This development is in perfect agreement with the infinite-product solution for Z at T_c and also with the latent heat, but does not have the factorisation property $Z=f(x)f(y)g(xy)$.

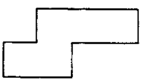
order to verify this functional equation we have introduced a new diagrammatic expansion corresponding to a partial resummation on one variable. One can remark that this resummation also holds in three dimensions (Jaekel and Maillard 1982). We have verified this functional equation up to fifth order and we have then outlined some of the uses of this inverse relation. The method of the inverse functional equation possesses several advantages. It can be easier to use than the traditional methods, as we have seen on the Potts model at T_c. It can be used for problems where the Bethe ansatz is not yet known, but for which a star-triangle relation exists—this is the case for Baxter's (1980a) hard-hexagon model—and it can be used for models where no star-triangle relation is known to exist. These advantages are still relevant when we consider higher-dimensional models. As far as the three-dimensional equivalent of the generalised star-triangle relation is concerned (the tetrahedron relation), it is still difficult to find any non-trivial solution. It is even less obvious to find an equivalent of the Bethe ansatz. In this context one can find inverse relations for various problems: one-dimensional problems (rubber bands, percolation, directed percolation), two-dimensional Ising or Potts models (even non-planar or with field), and three- and even higher-dimensional models (with or without field) with corresponding functional equations. Furthermore, the new diagrammatics described here which is relevant to the inverse relation also exists for the previously cited models (with the exception of directed percolation). One can then verify with these models the exactness of the functional equation for the partition function, and apply the methods developed here.

Acknowledgments

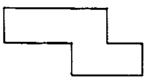
We want particularly to express our gratitude to M Gaudin and J L Verdier for numerous and fruitful discussions. It is a pleasure to thank J B Zuber for discussions and help in the puzzling counting of disconnected terms.

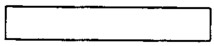
Appendix 1

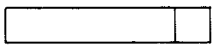
A unique diagram will represent all the diagrams which can be deduced from one another by symmetry. For instance

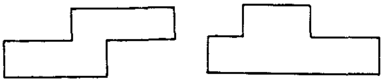


will represent itself and



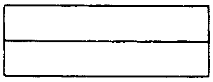
at order $\frac{1}{c^2}$:  $\frac{q-1}{c^2}\left(\frac{1}{b^2-1}\right)$

at order $\frac{1}{c^3}$:  $\frac{(q-1)(q-2)}{c^3}\left(\frac{1}{b^2-1}\right)^2$

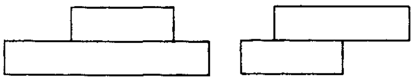
at order $\frac{1}{c^4}$:  $\frac{4(q-1)}{c^4}\left(\frac{1}{b^2-1}\right)^3$

 $\frac{4(q-1)}{c^4}\left(\frac{1}{b^2-1}\right)^2$  $\frac{q-1}{c^4}\left(\frac{1}{b^2-1}\right)$

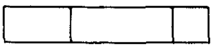
which sum to give $\frac{q-1}{c^4}\left(\frac{(b^2+1)^2}{(b^2-1)^3}\right)$

 $\frac{(q-1)(q-2)}{c^4}\left(\frac{1}{b^3-1}\right)$

 $\frac{4(q-1)(q-2)}{c^4}\left(\frac{1}{b^3-1}\right)\left(\frac{1}{b^2-1}\right)$

 $\frac{4(q-1)(q-2)}{c^4}\left(\frac{1}{b^3-1}\right)\left(\frac{1}{b^2-1}\right)^2$

summing to $\frac{(q-1)(q-2)}{c^4}\left(\frac{1}{b^3-1}\right)\left(\frac{b^2+1}{b^2-1}\right)^2$

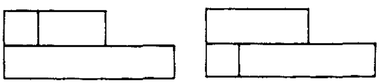
 $\frac{(q-1)(q-2)^2}{c^4}\left(\frac{1}{b^2-1}\right)^3$

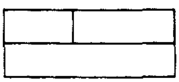
and at last the disconnected term

 $-\frac{(q-1)^2}{c^4}\left[\sum_{p,q=1}^{\infty}\left(\frac{1}{b^2}\right)^{p+q}\left(\frac{3(p+q)-1}{2}\right)\right]$

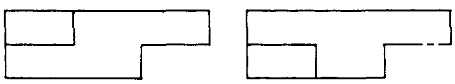
$$=\frac{(q-1)^2}{c^4}\left[\frac{1}{2}\left(\frac{1}{b^2-1}\right)^2-3\frac{b^2}{(b^2-1)^3}\right]$$

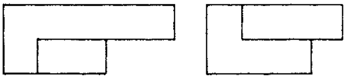
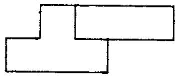
at order $\frac{1}{c^5}$:  $\frac{8(q-1)(q-2)(q-3)}{(b^3-1)^2(b^2-1)^2}$

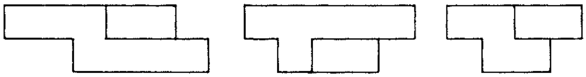
 $\frac{8(q-1)(q-2)(q-3)}{(b^3-1)^2}\left(\frac{1}{b^2-1}\right)$

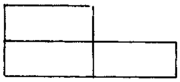
 $\frac{2(q-1)(q-2)(q-3)}{(b^3-1)^2}$

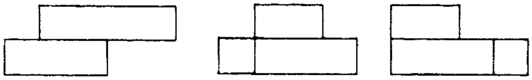
summing to $\frac{2(q-1)(q-2)(q-3)}{(b^3-1)^2}\left(\frac{b^2+1}{b^2-1}\right)^2$

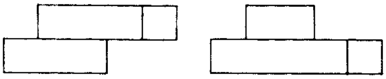
 $\frac{4(q-1)(q-2)}{(b^3-1)(b^2-1)}$ 

 $\dfrac{16(q-1)(q-2)}{(b^3-1)(b^2-1)^2}$ 

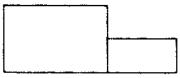
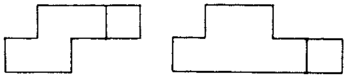
 $\dfrac{16(q-1)(q-2)}{(b^3-1)(b^2-1)^3}$

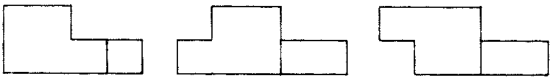
summing to $\dfrac{4(q-1)(q-2)}{(b^3-1)(b^2-1)}\left(\dfrac{b^2+1}{b^2-1}\right)^2$;  $\dfrac{4(q-1)(q-2)^2}{(b^3-1)(b^2-1)}$

 $\dfrac{12(q-1)(q-2)^2}{(b^2-1)^2(b^3-1)}$

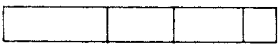
 $\dfrac{8(q-1)(q-2)^2}{(b^3-1)(b^2-1)^3}$

summing to $\dfrac{4(q-1)(q-2)^2b^2(b^2+1)}{(b^3-1)(b^2-1)^3}$

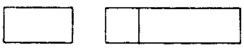
 $\dfrac{4(q-1)(q-2)}{(b^2-1)^2}$.  $\dfrac{8(q-1)(q-2)}{(b^2-1)^4}$

 $\dfrac{12(q-1)(q-2)}{(b^2-1)^3}$

summing to $\dfrac{4(q-1)(q-2)}{(b^2-1)^4}\,b^2(b^2+1)$

 $(q-1)(q-2)^3\left(\dfrac{1}{b^2-1}\right)^4$

and at last the disconnected term

 $(q-1)^2(q-2)\left[\sum_{p,q,r=1}^{\infty}[3(p+q+r)-1]\left(\dfrac{1}{b^2}\right)^{p+q+r}\right]$

$$=(q-1)^2(q-2)\left(\frac{1}{(b^2-1)^3}-9\frac{b^2}{(b^2-1)^4}\right).$$

Let us sum $\ln\Lambda(b,c)$ and $\ln\Lambda(1/b, 2-q-c)$:

$$\ln\Lambda(b,c)=\frac{q-1}{c^2}\left(\frac{1}{b^2-1}\right)+\frac{(q-1)(q-2)}{c^3}\left(\frac{1}{b^2-1}\right)^2+\cdots$$

$$\ln\Lambda\left(\frac{1}{b},2-q-c\right)=\frac{q-1}{c^2}\left(\frac{-b^2}{b^2-1}\right)\left[1+2\left(\frac{2-q}{c}\right)+\cdots\right]$$

$$-\frac{(q-1)(q-2)}{c^3}\left(\frac{-b^2}{b^2-1}\right)^2[1+\cdots]+\cdots.$$

This gives at order $1/c^2$

$$\left(\frac{q-1}{c^2}\right)\left(\frac{1}{b^2-1}\right)+\frac{q-1}{c^2}\left(\frac{-b^2}{b^2-1}\right)=-\frac{q-1}{c^2}$$

and at order $1/c^3$

$$\frac{(q-1)(q-2)}{c^3}\left(\frac{1}{b^2-1}\right)^2-\frac{(q-1)(q-2)}{c^3}\left(\frac{b^4}{(b^2-1)^2}\right)+\frac{2(q-1)(q-2)}{c^3}\left(\frac{b^2}{b^2-1}\right)$$
$$=\frac{(q-1)(q-2)}{c^3}.$$

For higher orders, as the number of terms is increasing rapidly we just sketch the calculation. At order $1/c^4$, consider first the terms with a b^3-1 singularity. Summing up the same diagrams in $\ln \Lambda(b, c)$ and $\ln \Lambda(1/b, 2-q-c)$

$$(q-1)(q-2)\left(\frac{1-b^3}{b^3-1}\right) \quad \text{and} \quad \frac{4(q-1)(q-2)}{(b^3-1)(b^2-1)^2}(b^2-b^5)$$

we see how the b^3-1 singularity disappears, and the same for the b^2-1 singularity as well, leaving a constant term

$$-\tfrac{1}{2}(q-1)^2-(q-1)(q-2)^2.$$

At order $1/c^5$, as before, the pole of order two for the b^3-1 singularity disappears:

$$\frac{2(q-1)(q-2)(q-3)}{(b^3-1)^2}\left(\frac{b^2+1}{b^2-1}\right)^2(1-b^6)=\frac{2(q-1)(q-2)(q-3)}{(1-b^3)}(1+b^{33})\left(\frac{1+b^2}{1-b^2}\right)^2.$$

Regrouping this term and the other simple poles at $b^3=1$, one obtains

$$2(q-1)^2(q-2)\left(\frac{b^2+1}{b^2-1}\right)^2-4(q-1)(q-2)\left(\frac{b^6+qb^4+(q-1)b^2}{(b^2-1)^3}\right)$$

so that the b^3-1 singularity vanishes. Adding the remaining terms leads to the constant

$$(q-1)(q-2)^3+(q-1)^2(q-2).$$

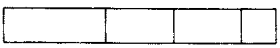
Comparing all these results with the expansion

$$\ln\left(1+\frac{q-2}{c}+\frac{1-q}{c^2}\right)-\ln\left(1+\frac{q-2}{c}\right)$$
$$=\frac{1-q}{c^2}+\frac{(q-1)(q-2)}{c^3}-\frac{1}{2}\frac{(q-1)^2}{c^4}+\frac{(q-2)^2(1-q)}{c^4}+\frac{(q-2)(1-q)^2}{c^5}$$
$$+\frac{(q-2)^3(q-1)}{c^5}+\cdots$$

one verifies that the functional equation is satisfied up to fifth order.

Appendix 2

Consider order $1/c^n$ and more precisely the diagrams of order $n-1$ in q: in the expansion of $\ln \Lambda(b, c)$ there is only one such diagram:



It has $n-1$ loops and n vertical bonds. Its contribution is

$$\frac{(q-1)(q-2)^{n-2}}{c^n(b^2-1)^{n-1}}.$$

The expansion of $\ln \Lambda(1/b, 2-q-c)$ includes the same diagram with contribution

$$\frac{(q-1)(q-2)^{n-2}}{(2-q-c)^n(b^2-1)^{n-1}}(-b^2)^{n-1}$$

or

$$\frac{(q-1)(q-2)^{n-2}}{(-1)^n c^n}\left(\frac{-b^2}{b^2-1}\right)^{n-1}$$

(at order $1/c^n$) and also contributions coming from the expansions of a similar term for $n=2, 3, \ldots, n-1$; for instance,

$$\frac{(q-1)(q-2)^{n-3}}{(2-q-c)^{n-1}}\left(\frac{-b^2}{b^2-1}\right)^{n-2}$$

gives the term

$$\frac{(q-1)(q-2)^{n-2}}{(-c)^n}C^1_{n-1}\left(\frac{-b^2}{b^2-1}\right)^{n-1}$$

and, in a more general way,

$$\frac{(q-1)(q-2)^{n-2}}{(-c)^n}C^p_{n-1}\left(\frac{-b^2}{b^2-1}\right)^{n-1-p}$$

is obtained from

$$\frac{(q-1)(q-2)^{n-2-p}}{(2-q-c)^{n-p}}\left(\frac{-b^2}{b^2-1}\right)^{n-1-p}.$$

The sum of all these contributions is

$$\begin{aligned}
&\frac{(q-1)(q-2)^{n-2}}{c^n}\left[\frac{1}{(b^2-1)^{n-1}}+\left(\frac{-b^2}{b^2-1}\right)^{n-1}+C^1_{n-1}\left(\frac{-b^2}{b^2-1}\right)^{n-2}+\cdots\right]\\
&\qquad=\frac{(q-1)(q-2)^{n-2}}{c^n}\left\{\frac{1}{(b^2-1)^{n-1}}+\left[\left(1-\frac{b^2}{b^2-1}\right)^{n-1}-1\right](-1)^n\right\}\\
&\qquad=\frac{(q-1)(q-2)^{n-2}(-1)^{n+1}}{c^n}.
\end{aligned}$$

On the other hand, the expansion of

$$\ln\left(1+\frac{q-2}{c}+\frac{1-q}{c^2}\right)-\ln\left(1+\frac{q-2}{c}\right)$$

gives us, at order $1/c^n$ and for the q^{n-1} term,

$$(-1)^{n+1}\frac{(q-1)(q-2)^{n-2}}{c^n}$$

Appendix 3

To illustrate the analytical inadequacy of a certain choice of variables, let us come back to the Ising model. In this case, S and I (symmetry and inverse) generate a finite group: $(\mathrm{SI})^4 = e$ (e is the identity). Consider the corresponding sequence of transformations

$$Z\left(\frac{1}{b}, -c\right) = \left(1 - \frac{1}{c^2}\right) / Z(b, c) \qquad (\mathrm{I})$$

$$Z\left(-c, \frac{1}{b}\right) = \left(1 - \frac{1}{c^2}\right) / Z(b, c) \qquad (\mathrm{SI})$$

$$Z\left(-\frac{1}{b}, -\frac{1}{c}\right) = \frac{1-b^2}{1-1/c^2} Z(b, c) \qquad (\mathrm{SI})^2$$

$$Z(b, c) = \frac{1-b^2}{1-1/c^2}\,\frac{1-1/b^2}{1-c^2} Z(b, c) \quad (\mathrm{SI})^4$$

This leads to an apparent contradiction which expresses the fact that Z possesses several determinations.

References

Baxter R J 1972 *Ann. Phys., NY* **70** 193
—— 1973a *Ann. Phys., NY* **71** 1
—— 1973b *J. Phys. C: Solid State Phys.* **6** L445
—— 1978 *Phil. Trans. R. Soc.* A **289** 315
—— 1980a *J. Phys. A: Math. Gen.* **13** L61
—— 1980b in *Fundamental Problems in Statistical Mechanics V: Proc. Enschede Summer School* ed E G D Cohen (Amsterdam: North Holland)
Baxter R J, Sykes M S and Watts M G 1975 *J. Phys. A: Math. Gen.* **8** 245
—— 1976 *J. Phys. A: Math. Gen.* **9** L149
Baxter R J, Temperley H N V and Ashley S E 1978 *Proc. R. Soc.* A **358** 535
Berg B and Weisz P H 1978 *Nucl. Phys.* B **146** 205
Bethe H A 1931 *Z. Phys.* **71** 205
Coxeter H S M and Moser W O J 1972 *Generators for Discrete Groups* 3rd edn (Berlin: Springer)
Gaudin M 1979 *Note* CEA-N-1559(2)
Ginsparg P, Goldschmitt Y Y and Zuber J B 1980 *Nucl. Phys. B* **170** 409
Jaekel M T and Maillard J M 1981 *Publication of IRMA, Strasbourg* vol 29, RCP 25
—— 1982 *J. Phys. A: Math. Gen.* **15** 1309
Kihara T, Midzuno Y and Shizume T 1954 *J. Phys. Soc. Japan* **9** 681
Kramers H A and Wannier G H 1941 *Phys. Rev.* **60** 252
Kulish P P and Sklyanin E K 1979 *LOMI preprint* E-3-1979
Magnus W, Oberhettinger F and Soni R P 1972 *Formulas and Theorems for the Special Functions of Mathematical Physics*
Potts R B 1952 *Proc. Camb. Phil Soc.* **48** 106
Schultz C L 1981 *Phys. Rev. Lett.* **10** 629
Sklyanin E K, Taktadjyan L A and Faddeev L D 1979 *Teor. Mat. Fiz* **40** 194
Stroganov Y G 1979 *Phys. Lett.* **74A** 116
Temperley H N V and Lieb E 1971 *Proc. R. Soc.* A **322** 251
Watson P G 1972 in *Phase Transitons and Critical Phenomena* vol 2, ed C Domb and M S Green (New York: Academic) p 113
Zamolodchikov A B 1979 *Commun. Math. Phys.* **69** 165
Zamolodchikov A B and Zamolodchikov A B 1979 *Ann. Phys., NY* **120** 253